Bessel’s Function

LECTURE 1

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Bessel's Equation

- Bessel Equation of order ν:

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right)y = 0$$

- Note that $x = 0$ is a regular singular point.
- Friedrich Wilhelm Bessel (1784 – 1846) studied disturbances in planetary motion, which led him in 1824 to make the first systematic analysis of solutions of this equation. The solutions became known as Bessel functions.
- In this section, we study the following cases:
  - Bessel Equations of order zero: $\nu = 0$
  - Bessel Equations of order one-half: $\nu = ½$
  - Bessel Equations of order one: $\nu = 1$

Bessel Equation of Order Zero

- The Bessel Equation of order zero is

$$x^2 y'' + xy' + x^2 y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0,\ x > 0$$

- Taking derivatives,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad y'(x) = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1},$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$$

- Substituting these into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

Indicial Equation

- From the previous slide,

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

- Rewriting,

$$a_0\left[r(r-1)+r\right]x^r + a_1\left[(r+1)r+(r+1)\right]x^{r+1}$$

$$+\sum_{n=2}^{\infty}\left\{a_n\left[(r+n)(r+n-1)+(r+n)\right]+a_{n-2}\right\}x^{r+n} = 0$$

- or

$$a_0 r^2 x^r + a_1 (r+1)^2 x^{r+1} + \sum_{n=2}^{\infty}\left\{a_n (r+n)^2 + a_{n-2}\right\}x^{r+n} = 0$$

- The indicial equation is $r^2 = 0$, and hence $r_1 = r_2 = 0$.